CHAPTER

11

PARALLELOGRAMS AND TRIANGLES

Animation 11.1: Triangle to Square
Source & Credit: takayaiwamoto

Students Learning Outcomes

After studying this unit, the students will be able to:

- prove that in a parallelogram
  - the opposite sides are congruent,
  - the opposite angles are congruent,
  - the diagonals bisect each other.
- prove that if two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.
- prove that the line segment, joining the midpoints of two sides of a triangle, is parallel to the third side and is equal to one half of its length.
- prove that the medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.
- prove that if three or more parallel lines make congruent segments on a transversal, they also intercept congruent segments on any other line that cuts them.

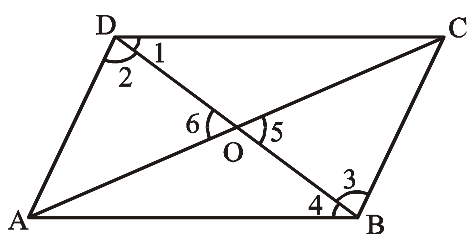
Introduction

Before proceeding to prove the theorems in this unit the students are advised to recall definitions of polygons like parallelogram, rectangle, square, rhombus, trapezium etc. and in particular triangles and their congruency.

Theorem 11.1.1

In a parallelogram

(i) Opposite sides are congruent.
(ii) Opposite angles are congruent.
(iii) The diagonals bisect each other.



Given

In a quadrilateral ABCD, $\overline{AB} \parallel \overline{DC}$, $\overline{BC} \parallel \overline{AD}$ and the diagonals $\overline{AC}$, $\overline{BD}$ meet each other at point O.

To Prove

(i) $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$
(ii) $\angle ADC \cong \angle ABC$, $\angle BAD \cong \angle BCD$
(iii) $\overline{OA} \cong \overline{OC}$, $\overline{OB} \cong \overline{OD}$

Construction

In the figure as shown, we label the angles as $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$, and $\angle 6$

Proof

Statements	Reasons
In $\Delta ABD \longleftrightarrow \Delta CDB$	
$\angle 4 \cong \angle 1$	alternate angles
$BD \cong BD$	Common
$\angle 2 \cong \angle 3$	alternate angles
$\therefore \Delta ABD \cong \Delta CDB$	A.S.A. ≅ A.S.A.
So, $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$	(corresponding sides of congruent triangles)
and $\angle A \cong \angle C$	(corresponding angles of congruent triangles)
(ii) Since	
$\angle 1 \cong \angle 4$ ...... (a)	Proved
and $\angle 2 \cong \angle 3$ ...... (b)	Proved
$\therefore m\angle 1 + m\angle 2 = m\angle 4 + m\angle 3$	from (a) and (b)
or $m\angle ADC \cong m\angle ABC$	Proved in (i)
$\angle ADC \cong \angle ABC$	
and	
$\angle BAD \cong \angle BCD$	
(iii) In $\Delta BOC \longleftrightarrow \Delta DOA$	
$\overline{BC} \cong \overline{AD}$	Proved in (i)
$\angle 5 \cong \angle 6$	vertical angles
$\angle 3 \cong \angle 2$	Proved
$\therefore \Delta BOC \cong \Delta DOA$	(A.A.S. ≅ A. A. S.)
Hence $\overline{OC} \cong \overline{OA}$, $\overline{OB} \cong \overline{OD}$	(corresponding sides of congruent triangles)

Corollary

Each diagonal of a parallelogram bisects it into two congruent triangles.

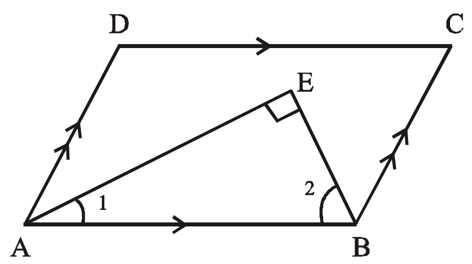
Example

The bisectors of two angles on the same side of a parallelogram cut each other at right angles.

Given

A parallelogram ABCD, in which $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$.

The bisectors of $\angle A$ and $\angle B$ cut each other at E.



To Prove

$m\angle E = 90^0$

Construction

Name the angles $\angle 1$ and $\angle 2$ as shown in the figure.

Proof

Statements	Reasons
$m\angle 1 + m\angle 2$	
$= \frac{1}{2}(m\angle BAD + m\angle ABC)$	$\left\{ \begin{array}{l} m\angle 1 = m\ \frac{1}{2}\angle BAD, \\ m\angle 2 = m\ \frac{1}{2}\angle ABC \end{array} \right.$
$= \frac{1}{2}(180^0)$	Int. angles on the same side of $\overline{AB}$ which cuts $\parallel$ segments $\overline{AD}$ and $\overline{BC}$ are supplementary.
$=90^0$	
Hence in ΔABE, $m\angle E = 90^0$	$m\angle 1 + m\angle 2 = 90^0$ (Proved)

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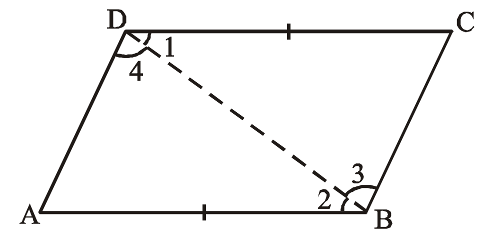
EXERCISE 11.1

1. One angle of a parallelogram is 130^0. Find the measures of its remaining angles.
2. One exterior angle formed on producing one side of a parallelogram is 40^0. Find the measures of its interior angles.

Theorem 11.1.2

If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.



Given

In a quadrilateral ABCD,
$\overline{AB} \cong \overline{DC}$ and $\overline{AB} \parallel \overline{DC}$

To Prove

ABCD is a parallelogram.

Construction

Join the point B to D and in the figure, name the angles as indicated: $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$

Proof

Statements	Reasons
In $\Delta ABD \longleftrightarrow \Delta CDB$	
$\overline{AB} \cong \overline{DC}$,	given
$\angle 2 \cong \angle 1$	alternate angles
$\overline{BD} \cong \overline{BD}$	Common
$\therefore$ $\Delta ABD \cong \Delta CDB$	S.A.S. postulate
Now $\angle 4 \cong \angle 3$ ....(i)	(corresponding angles of congruent triangles)
$\therefore$ $\overline{AD} \parallel \overline{BC}$ ....(ii)	from (i)
and $\overline{AD} = \overline{BC}$ ....(iii)	corresponding sides of congruent $\triangle$s

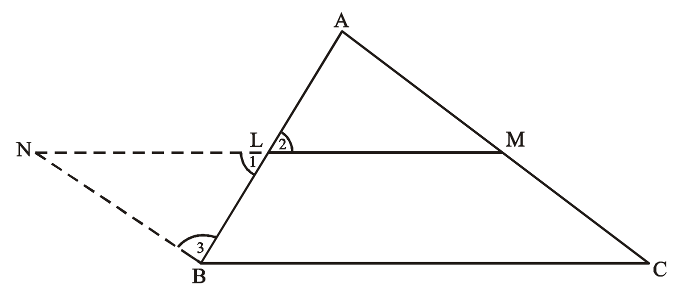
Also $\overline{AB} \parallel \overline{DC}$ ....(iv)	given
Hence ABCD is a parallelogram	from (ii) – (iv)

EXERCISE 11.2

1. Prove that a quadrilateral is a parallelogram if its
   (a) opposite angles are congruent. (b) diagonals bisect each other.
2. Prove that a quadrilateral is a parallelogram if its opposite sides are congruent.

Theorem 11.1.3

The line segment, joining the mid-points of two sides of a triangle, is parallel to the third side and is equal to one half of its length.



Given

In ΔABC, The mid-points of $\overline{AB}$ and $\overline{AC}$ are L and M respectively.

To Prove

$\overline{LM} \parallel \overline{BC}$ and $m\overline{LM} = \frac{1}{2} m\overline{BC}$

Construction

Join M to L and produce $\overline{ML}$ to N such that $\overline{ML} \cong \overline{LN}$.
Join N to B and in the figure, name the angles as $\angle 1, \angle 2$ and $\angle 3$ as shown.

Proof

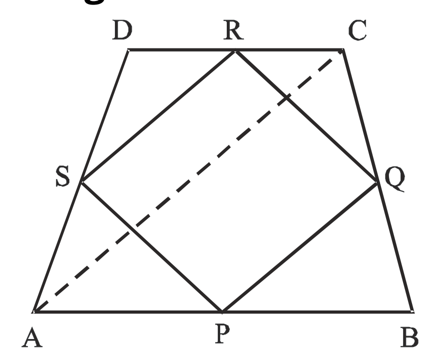
Statements	Reasons
In $\Delta BLN \longleftrightarrow \Delta ALM$	
$\overline{BL} \cong \overline{AL}$	Given
$\angle 1 \cong \angle 2$	vertical angles
$\overline{NL} \cong \overline{ML}$	Construction
$\therefore$ $\Delta BLN \cong \Delta ALM$	S.A.S. postulate
$\therefore$ $\angle A \cong \angle 3$ ....(i)	(corresponding angles of congruent triangles)
and $\overline{NB} \cong \overline{AM}$ ....(ii)	(corresponding sides of congruent triangles)
But $\overline{NB} \parallel \overline{AM}$	From (i), alternate $\angle$s
Thus $\overline{NB} \parallel \overline{MC}$ ....(iii)	(M is a point of $\overline{AC}$)
$\overline{MC} \cong \overline{AM}$ ....(iv)	Given
$\overline{NB} \cong \overline{MC}$ ....(v)	{from (ii) and (iv)}
$\therefore$ BCMN is a parallelogram	from (iii) and (v)
$\therefore$ $\overline{BC} \parallel \overline{LM}$ or $\overline{BC} \parallel \overline{NL}$	(opposite sides of a parallelogram BCMN)
$\overline{BC} \cong \overline{NM}$ ....(vi)	(opposite sides of a parallelogram)
$m\overline{LM} = m\frac{1}{2}\overline{NM}$ ....(vii)	Construction
Hence $m\overline{LM} = \frac{1}{2} m\overline{BC}$	{from (vi) and (vii)}

Note that instead of producing $\overline{ML}$ to N, we can take N on $\overline{LM}$ produced.

Example

The line segments, joining the mid-points of the sides of a quadrilateral, taken in order, form a parallelogram.

Given



A quadrilateral ABCD, in which P is the mid-point of $\overline{AB}$, Q is the mid-point of $\overline{BC}$, R is the mid-point of $\overline{CD}$, S is the mid-point of $\overline{DA}$.
P is joined to Q, Q is joined to R.
R is joined to S and S is joined to P.

To Prove

PQRS is a parallelogram.

Construction

Join A to C.

Proof

Statements	Reasons
In ΔDAC,	
$\overline{SR} \parallel \overline{AC}$ $m\overline{SR} = m\frac{1}{2}\overline{AC}$	S is the midpoint of $\overline{DA}$ R is the midpoint of $\overline{CD}$
In ΔBAC,	
$\overline{PQ} \parallel \overline{AC}$ $m\overline{PQ} = m\frac{1}{2}\overline{AC}$	P is the midpoint of $\overline{AB}$ Q is the mid-point of $\overline{BC}$
$\overline{SR} \parallel \overline{PQ}$	Each $\parallel \overline{AC}$
$m\overline{SR} = m\overline{PQ}$	Each $= m\frac{1}{2}\overline{AC}$
Thus PQRS is a parallelogram	$\overline{SR} \parallel \overline{PQ}$, $m\overline{SR} = m\overline{PQ}$ (proved)

EXERCISE 11.3

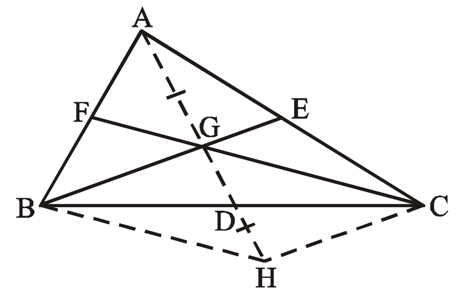
1. Prove that the line-segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.
2. Prove that the line-segments joining the midpoints of the opposite sides of a rectangle are the right-bisectors of each other.
   [**Hint**: Diagonals of a rectangle are congruent.]
3. Prove that the line-segment passing through the midpoint of one side and parallel to another side of a triangle also bisects the third side.

Theorem 11.1.4

The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.

Given

$\triangle ABC$



To Prove

The medians of the $\triangle ABC$ are concurrent and the point of concurrency is the point of trisection of each median.

Construction

Draw two medians $\overline{BE}$ and $\overline{CF}$ of the $\triangle ABC$ which intersect each other at G. Join A to G and produce it to point H such that $AG \cong \overline{GH}$. Join H to the points B and C.

$\overline{AH}$ intersects $\overline{BC}$ at the point D.

Proof

Statements	Reasons
In $\triangle ACH$, $\overline{GE} \parallel \overline{HC}$	G and E are mid-points of sides AH and AC respectively
or $\overline{BE} \parallel \overline{HC}$ ....(i)	G is a point of BE
Similarly, $\overline{CF} \parallel \overline{HB}$ ....(ii)	
$\therefore$ BHCG is a parallelogram	from (i) and (ii)
and $m\overline{GD} = mGH$ ....(iii)	(diagonals $\overline{BC}$ and $\overline{GH}$ of a parallelogram BHCG intersect each other at point D)
$\overline{BD} \cong \overline{CD}$ $\overline{AD}$ is a meadian of $\triangle ABC$	
Meadians $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ pass through the point G	(G is the intersecting point of $\overline{BE}$ and $\overline{CF}$ and $\overline{AD}$ pass through it.)
Now $\overline{GH} \cong \overline{AG}$ ....(iv)	construction

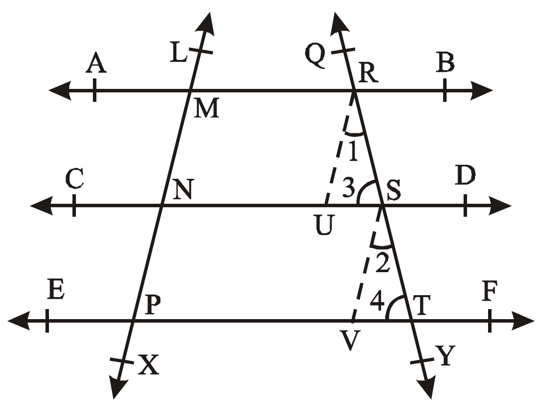
$\therefore \quad m\overline{GD} = \frac{1}{2}\, m\overline{AG}$ and G is the point of trisection of $\overline{AD}$ ....(v) Similarly it can be proved that G is also the point of trisection of $\overline{CF}$ and $\overline{BE}$	from (iii) and (iv)

EXERCISE 11.4

1. The distances of the points of concurrency of the median of a triangle from its vertices are respectively 1.2 cm, 1.4 cm and 1.6 cm. Find the lengths of its medians.
2. Prove that the point of concurrency of the medians of a triangle and the triangle which is made by joining the mid-point of its sides is the same.

Theorem 11.1.5

If three or more parallel lines make congruent segments on a transversal, they also intercept congruent segments on any other line that cuts them.



Given

$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$

The transversal LX intersects $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ at the points M, N and P respectively, such that $\overline{MN} \cong \overline{NP}$. The transversal $\overleftrightarrow{QY}$ intersects them at points R, S and T respectively.

To Prove

$\overline{RS} \cong \overline{ST}$

Construction

From R, draw $\overline{RU} \parallel \overline{LX}$, which meets $\overline{CD}$ at U. From S, draw $\overline{SV} \parallel \overline{LX}$ which meets $\overline{EF}$ at V. As shown in the figure let the angles be labelled as

$\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$

Proof

Statements	Reasons
MNUR is a parallelogram	$\overline{RU} \parallel \overline{LX}$ (construction) $\overline{AB} \parallel \overline{CD}$ (given)
$\therefore \quad \overline{MN} \cong \overline{RU}$ (i)	(opposite sides of a parallelogram)
Similarly, $\overline{NP} \cong \overline{SV}$ (ii)	
But $\quad \overline{MN} \cong \overline{NP}$ (iii)	Given
$\therefore \quad \overline{RU} \cong \overline{SV}$	{from (i), (ii) and (iii)}
Also $\quad \overline{RU} \parallel \overline{SV}$	each $\parallel \overleftrightarrow{LX}$ (construction)
$\therefore \quad \angle 1 \cong \angle 2$	Corresponding angles
and $\quad \angle 3 \cong \angle 4$	Corresponding angles
In $\quad \Delta RUS \longleftrightarrow \Delta SVT$,	
$\overline{RU} \cong \overline{SV}$	Proved
$\angle 1 \cong \angle 2$	Proved
$\angle 3 \cong \angle 4$	Proved
$\therefore \quad \Delta RUS \cong \Delta SVT$	S.A.A. $\cong$ S.A.A.
Hence $\quad \overline{RS} \cong \overline{ST}$	(corresponding sides of congruent triangles)

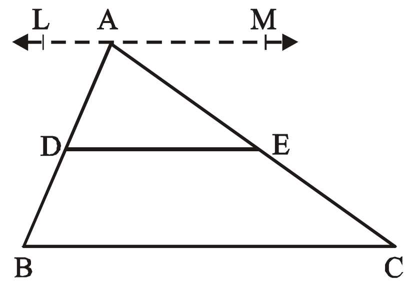
Note: This theorem helps us in dividing line segment into parts of equal lengths. It is also used in the division of a line segment into proportional parts.

Corollaries

(i) A line, through the mid-point of one side, parallel to another side of a triangle, bisects the third side.

Given

In ΔABC, D is the mid-point of $\overline{AB}$.
$\overline{DE} \parallel \overline{BC}$ which cuts $\overline{AC}$ at E.



To Prove

$\overline{AE} \cong \overline{EC}$

Construction

Through A, draw $\overleftrightarrow{LM} \parallel \overline{BC}$.

Proof

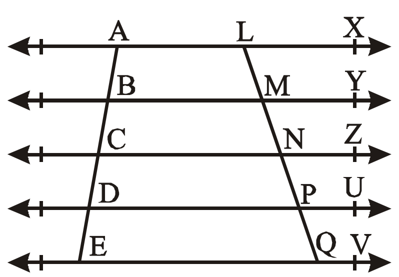
Statements	Reasons
Intercepts cut by $\overleftrightarrow{LM}$, $\overline{DE}$, $\overline{BC}$ on $\overline{AC}$ are congruent. i.e., $\overline{AE} \cong \overline{EC}$.	$\{$ Intercepts cut by parallels $\overleftrightarrow{LM}$, $\overline{DE}$, $\overline{BC}$ on $\overline{AB}$ are congruent (given)

(ii) The parallel line from the mid-point of one non-parallel side of a trapezium to the parallel sides bisects the other non-parallel side.

(iii) If one side of a triangle is divided into congruent segments, the line drawn from the point of division parallel to the other side will make congruent segments on third side.

EXERCISE 11.5

1. In the given figure
$\overleftrightarrow{AX} \parallel \overleftrightarrow{BY} \parallel \overleftrightarrow{CZ} \parallel \overleftrightarrow{DU} \parallel \overleftrightarrow{EV}$
and $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE}$
If $m\overline{MN}$ = 1cm, then find the length of $\overline{LN}$ and $\overline{LQ}$.



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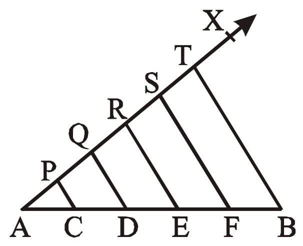
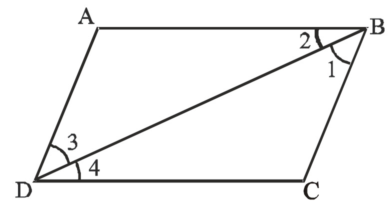
2. Take a line segment of length 5.5 cm and divide it into five congruent parts.
[Hint: Draw an acute angle $\angle BAX$ on $\overline{AX}$ take $\overline{AP} \cong \overline{PQ} \cong \overline{QR} \cong \overline{RS} \cong \overline{ST}$.
Join T to B. Draw lines parallel to $\overline{TB}$ from the points P, Q, R and S.]

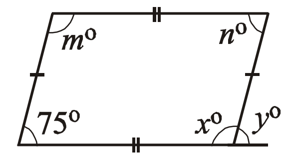


REVIEW EXERCISE 11

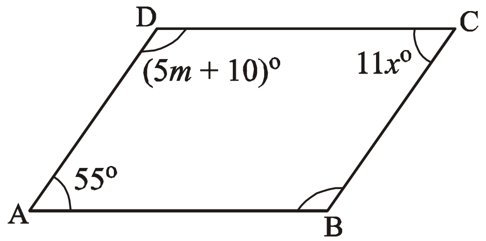
1. Fill in the blanks.
   (i) In a parallelogram opposite sides are ....
   (ii) In a parallelogram opposite angles are ....
   (iii) Diagonals of a parallelogram .... each other at a point.
   (iv) Medians of a triangle are ....
   (v) Diagonal of a parallelogram divides the parallelogram into two .... triangles
2. In parallelogram ABCD
   (i) $m\overline{AB}$.......$m\overline{DC}$ (ii) $m\overline{BC}$........$m\overline{AD}$
   (iii) $m\angle 1 \cong$ .... (iv) $m\angle 2 \cong$ ....



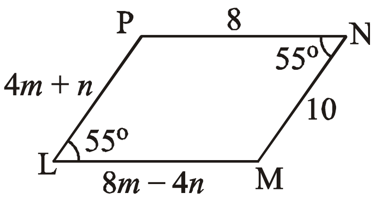
3. Find the unknowns in the given figure.



4. If the given figure ABCD is a parallelogram, then find x, m.



5. The given figure LMNP is a parallelogram. Find the value of m, n.



6. In the question 5, sum of the opposite angles of the parallelogram is 110^0, find the remaining angles.

SUMMARY

In this unit we discussed the following theorems and used them to solve some exercises. They are supplemented by unsolved exercises to enhance applicative skills of the students.

- In a parallelogram
  - (i) Opposite sides are congruent.
  - (ii) Opposite angles are congruent.
  - (iii) The diagonals bisect each other.
- If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.
- The line segment, joining the mid-points of two sides of a triangle, is parallel to the third side and is equal to one half of its length.
- The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.
- If three or more parallel lines make congruent segments on a transversal, they also intercept congruent segments on any other line that cuts them.